Chapter 9  Simple Harmonic Motion

Curriculum Specification

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9.1  Kinematics of Simple Harmonic Motion

a) Explain SHM. (C1, C2)

b) Solve problem related to SHM displacement equation, \( y = A \sin \omega t \) (C3, C4)

c) Derive equations:
   i. velocity,
   \[ v = \frac{dy}{dt} = \pm \omega \sqrt{A^2 - y^2} \]
   ii. acceleration,
   \[ a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -\omega^2 y \]
   iii. kinetic energy,
   \[ K = \frac{1}{2} m \omega^2 (A^2 - y^2) \]
   iv. potential energy,
   \[ U = \frac{1}{2} m \omega^2 y^2 \]

(C1, C2)

d) Emphasise the relationship between total SHM energy and amplitude. (C1, C2)

e) Apply velocity, acceleration, kinetic energy and potential energy for SHM. (C3, C4)

9.2  Graphs of Simple Harmonic Motion

a) Discuss the following graphs:
   i. displacement-time
   ii. velocity-time
   iii. acceleration-time
   iv. energy-displacement

(C1, C2)

9.3  Period of Simple Harmonic Motion

a) Use expression for period of SHM, \( T \) for simple pendulum and single spring (C3, C4)

b) Determine the acceleration, \( g \) due to gravity using simple pendulum. (Experiment 5: SHM) (C1, C2, C3, C4)

c) Investigate the effect of large amplitude oscillation to the accuracy of acceleration due to gravity, \( g \) obtained from the experiment. (Experiment 5: SHM) (C1, C2, C3, C4)
9.1 Kinematics of Simple Harmonic Motion

- Any oscillating system for which the net restoring force is directly proportional to the negative of the displacement \( F = -ky \) is said to exhibit simple harmonic motion (SHM).

**Example**

- Applying Newton’s second law:
  \[
  \sum F = ma \Rightarrow -kx = ma
  \]
  \[
  a = -\frac{k}{m}y
  \]

- That is, an object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium.
  \[
  a \propto -y
  \]

- The equilibrium position is a position at which the body would come to rest.
- The maximum excursion from equilibrium is the amplitude \( A \) of the motion.
- When an object moves in simple harmonic motion, a graph of its position as a function of time has a sinusoidal shape with amplitude \( A \).
Displacement

Consider a complete round-trip in SHM: O → A → O → −A → O (1 cycle)

From the reference circle, the displacement of the object from equilibrium:

\[ y = A \sin \theta \]

where \( \theta \) is in radians.

The object in the reference circle is rotating with uniform angular velocity \( \omega \), we can write \( \theta = \omega t \).

\[ y = A \sin(\omega t) \]

Phase

In general, at \( t = 0, y = 0 \). Motion starts from equilibrium.

If the motion is NOT start from equilibrium (at \( t = 0, y \neq 0 \)), then the equation should be written as

\[ y = A \sin(\omega t \pm \phi) \]
Example

**Velocity**

Equation for linear velocity of an object undergoing SHM can be derived by differentiating equation of displacement, with respect to time:

\[ v = \frac{dy}{dt} \quad \Rightarrow \quad v = \frac{d}{dt}(A \sin \omega t) \]

**Using substitution method**

Assume \( u = \omega t \), \( y = A \sin u \) \( \Rightarrow \frac{du}{dt} = \omega \), \( \frac{dy}{du} = A \cos u \)

\[ v = \frac{dy}{dt} = \frac{du}{dt} \times \frac{dy}{du} \quad \Rightarrow \quad v = A\omega \cos \omega t \quad \text{Velocity in term of time} \ t \]

From equation \( y = A \sin \omega t \),

\[ \sin \omega t = \frac{y}{A} \quad \Rightarrow \quad \sin^2 = \frac{y^2}{A^2} \quad (1) \]

Thus,

\[ v = A\omega \cos \omega t \quad \Rightarrow \quad v^2 = A^2\omega^2\cos^2\omega t \quad (2) \]

From trigonometry identity,

\[ \cos^2\omega t + \sin^2\omega t = 1 \quad \Rightarrow \quad \cos^2\omega t = 1 - \sin^2\omega t \quad (3) \]

Substitute (2) into (1),

\[ v^2 = A^2\omega^2(1 - \sin^2\omega t) \quad (4) \]

Substitute (1) into (4),

\[ v^2 = A^2\omega^2 \left( 1 - \frac{y^2}{A^2} \right) \]

\[ v^2 = A^2\omega^2 \left( \frac{A^2 - y^2}{A^2} \right) \]

\[ v^2 = \omega^2(A^2 - y^2) \]

\[ v = \pm \sqrt{\omega^2(A^2 - y^2)} \]

Velocity in term of displacement \( y \)
**Acceleration**
Equation for linear acceleration of an object undergoing SHM can be derived by differentiating equation of velocity, with respect to time:

\[ a = \frac{dv}{dt} \Rightarrow a = \frac{d}{dt} (A \omega \cos \omega t) \]

\[ a = -A\omega^2 \sin \omega t \quad \text{Acceleration in term of time } t \]

Since \( y = A \sin \omega t \),

\[ a = -\omega^2 y \quad \text{Acceleration in term of displacement } x \]

**Kinetic Energy**
From kinetic energy equation: \( K = \frac{1}{2} mv^2 \)

Substitute \( v = A \omega \cos \omega t \),

\[ K = \frac{1}{2} m\omega^2 A^2 \cos^2 \omega t \quad \text{KE in term of time } t \]

Substitute \( v = \pm \omega \sqrt{A^2 - y^2} \),

\[ K = \frac{1}{2} m\omega (A^2 - y^2) \quad \text{KE in term of displacement } x \]

**Potential Energy**
From potential energy equation: \( U = \frac{1}{2} ky^2 \)

The equations for acceleration: \( a = -\frac{k}{m} y \) and \( a = -\omega^2 y \) \( \Rightarrow \) \( k = m\omega^2 \)

Thus,

\[ U = \frac{1}{2} m\omega^2 y^2 \quad \text{PE in term of displacement } x \]

Substitute \( y = A \sin \omega t \),

\[ U = \frac{1}{2} kA^2 \sin^2 \omega t \quad \text{PE in term of time } t \]

**Total Energy**

\[ E = K + U \]

\[ E = \frac{1}{2} m\omega^2 (A^2 - y^2) + \frac{1}{2} m\omega^2 y^2 \]

\[ E = \frac{1}{2} m\omega^2 A^2 \quad \text{OR} \quad E = \frac{1}{2} kA^2 \]
The total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude. Figure below shows the relationship between the total energy and the amplitude of the simple harmonic motion.

9.2 Graphs of Simple Harmonic Motion

**Displacement-time Graph**
From the general equation of displacement as a function of time:

\[ y = A \sin \omega t \]

**Velocity-time Graph**
From the general equation of velocity as a function of time in SHM:

\[ v = A \omega \cos \omega t \]
**Acceleration-time Graph**

From the general equation of acceleration as a function of time:

\[ a = -A\omega^2 \sin \omega t \]

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**How to Sketch Graph?**

**Step 1:** Compare the equation given with the general equation

\[ y = A \sin \omega t \quad v = A\omega \cos \omega t \quad a = -A\omega^2 \sin \omega t \]

From comparison, identify

- Amplitude, \( A \)
- Angular frequency, \( \omega \)
- Initial phase angle, \( \phi \)

Then determine period \( T \) by using \( \omega \):

\[ T = \frac{2\pi}{\omega} \]

**Step 2:** Draw the \( y \) and \( x \) axis with the proper values and units

**Step 3:** Sketch the graph (If sine, start from zero. If cosine, start from amplitude)

**Step 4:** Shift according to the initial phase angle

<table>
<thead>
<tr>
<th>Rules to follow when shifting</th>
<th>Value of ( \phi )</th>
<th>Shift ( y )-axis</th>
<th>Shift Graph</th>
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<tbody>
<tr>
<td>Positive</td>
<td>To the right</td>
<td>To the left</td>
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<tr>
<td>Negative</td>
<td>To the left</td>
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Example
Sketch the $x$ against $t$ graph for the following expression:

$$y = 2 \text{ cm} \sin \left(2\pi t + \frac{\pi}{2}\right)$$

Solution:
**Step 1:** Compare with the general equation

$$y = 2 \text{ cm} \sin \left(2\pi t + \frac{\pi}{2}\right)$$

$$y = A \sin(\omega t + \phi)$$

From comparison, identify

- Amplitude, $A$ : 2 cm
- Angular frequency, $\omega$ : $2\pi$ rad
- Initial phase angle, $\phi$ : $+\frac{\pi}{2}$ rad = $\frac{1}{4}$ revolution

Period, $T = \frac{2\pi}{\omega} = 1$ s

**Step 2:** Draw the $y$ and $x$ axis with the proper values and units

**Step 3:** Sketch the graph → sine function

**Step 4:** Shift according to the initial phase angle

Initial phase angle, $\phi = +\frac{\pi}{2}$ rad = $\frac{1}{4}$ rev

Since it is a positive value:

- Shift $y$ - axis to the right, OR
- Shift graph to the left
Summary: Graphical representation of simple harmonic motion

- **Energy-displacement Graph**
  - From the equations of **kinetic energy** as a term of displacement:
    \[
    K = \frac{1}{2} m\omega^2 (A^2 - y^2)
    \]
  - From the equations of **potential energy** as a term of displacement:
    \[
    U = \frac{1}{2} m\omega^2 y^2
    \]
  - From the equations of **total energy** as a term of displacement:
    \[
    E = \frac{1}{2} kA^2
    \]
    \[
    E = \frac{1}{2} m\omega^2 A^2
    \]

- At maximum amplitude, 
  - velocity is equal to zero
  - acceleration is maximum
- At equilibrium, 
  - velocity is maximum
  - acceleration is equal to zero
- Acceleration is always oppositely directed to the displacement from equilibrium.
Additional Knowledge: \(v-y\) graph, \(a-y\) graph and \(E-t\) graph

**Velocity-displacement graph**
From the general equation of velocity as a function of displacement:

\[
v = \pm \omega \sqrt{(A^2 - y^2)}
\]

**Acceleration-displacement graph**
From the general equation of acceleration as a function of displacement:

\[
a = -\omega^2 y
\]

**Energy-time graph**
From the equations of total energy as a term of time:

\[
E = \frac{1}{2} kA^2
\]

\[
E = \frac{1}{2} m\omega^2 A^2
\]

9.3 **Period of Simple Harmonic Motion**

**Simple Pendulum**
The component tangential to the circular path, \(mg \sin \theta\) is the **restoring force** which acted on the bob to bring it back to its equilibrium position.

\[
F = -mg \sin \theta
\]

Negative because it is oppositely directed to the displacement

Assume the angle \(\theta\) is small (\(0^\circ < 1^\circ\))

\[
\sin \theta \approx \theta \approx \frac{y}{l}
\]

Thus,

\[
F = -mg \left(\frac{y}{l}\right)
\]
Applying Newton’s second law:
\[ \sum F = ma \quad \implies \quad -mg \left( \frac{y}{l} \right) = ma \]
\[ a = -\left( \frac{g}{l} \right) y \]
\[ a \propto -y \quad \text{Pendulum exhibits SHM} \]

Recall and compare condition for any SHM: \( a = -\omega^2 y \)

Thus, \( \omega^2 = \frac{g}{l} \quad \implies \quad \omega = \frac{\sqrt{g}}{\sqrt{l}} \)

From \( \omega = \frac{2\pi}{T} \quad \implies \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} \)

The conditions for the simple pendulum executes SHM are
- the angle, \( \theta \) has to be small (less than 10°)
- the string has to be inelastic and light
- only the gravitational force and tension in the string acting on the simple pendulum

**Horizontal Spring Oscillation (Single Spring)**
Assume that the object is oscillating on frictionless surface; the only force acting on it is the **restoring force of the spring**.

\[ F = -ky \]

Applying Newton’s second law
\[ \sum F = ma \quad \implies \quad -ky = ma \]
\[ a = -\left( \frac{k}{m} \right) y \]
\[ a \propto -y \quad \text{Object is moving in SHM} \]

Recall and compare condition for any SHM: \( a = -\omega^2 y \)

Thus, \( \omega^2 = \frac{k}{m} \quad \implies \quad \omega = \sqrt{\frac{k}{m}} \)

From \( \omega = \frac{2\pi}{T} \quad \implies \quad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \)
Vertical Spring Oscillation (Single Spring)

A free light spring with spring constant, k, hung vertically, Figure (a).

An object of mass, m is tied to the lower end of the spring as shown in Figure (b). When the object achieves an equilibrium condition

$$\sum F = 0 \implies F_s - mg = 0$$

$$-ky_0 - mg = 0$$

$$-ky_0 = mg$$

The object is then pulled downwards to a distance, x and released as shown in Figure (c). Hence,

$$\sum F = ma \implies F_s - mg = ma$$

$$-k(y_0 + y) - (-ky_0) = ma$$

$$a = -\left(\frac{k}{m}\right)y$$

$$a \propto -y \quad \text{Object is moving in SHM}$$

Recall and compare condition for any SHM: $$a = -\omega^2 y$$

Thus, \(\omega^2 = \frac{k}{m}\) \(\implies \omega = \sqrt{\frac{k}{m}}\)

From \(\omega = \frac{2\pi}{T}\) \(\implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}\)

The conditions for the spring-mass system executes SHM are
- the elastic limit of the spring is not exceeded when the spring is being pulled
- the spring is light and obeys Hooke’s law
- no air resistance and surface friction
### Exercise

**Kinematics of Simple Harmonic Motion**

1. The displacement of an object is described by the following equation, where $y$ is in meters and $t$ is in seconds:

   \[ y = (0.30 \text{ m}) \cos(8.0t) \]

   Determine the oscillating object’s
   a. amplitude
   b. frequency
   c. period
   d. maximum speed
   e. maximum acceleration

2. A piston in a gasoline engine is in simple harmonic motion. The engine is running at the rate of 3 600 rev/min. Taking the extremes of its position relative to its center point as 65.00 cm, find the magnitudes of the
   a. maximum velocity
   b. maximum acceleration of the piston

3. A 0.500 kg cart connected to a light spring for which the force constant is 20.0 N m$^{-1}$ oscillates on a frictionless, horizontal air track.
   a. Calculate the maximum speed of the cart if the amplitude of the motion is 3.00 cm
   b. What is the velocity of the cart when the position is 2.00 cm?
   c. Compute the kinetic and potential energies of the system when the position of the cart is 2.00 cm.

4. A 0.25 kg mass at the end of a spring oscillates 2.2 times per second with an amplitude of 0.15 m. Determine
   a. the speed when it passes the equilibrium point
   b. the speed when it is 0.10 m from equilibrium
   c. the total energy of the system
   d. the equation describing the motion of the mass, assuming that at $x$ was a maximum

5. A 3.2 kg block is hanging stationary from the end of a vertical spring that is attached to the ceiling. The elastic potential energy of this spring-block system is 1.8 J. What is the elastic potential energy of the system when the 3.2 kg block is replaced by a 5.0 kg block?

6. A vertical spring with a spring constant of 450 N m$^{-1}$ is mounted on the floor. From directly above the spring, which is unstrained, a 0.30 kg block is dropped from rest. It collides with and sticks to the spring, which is compressed by 2.5 cm in bringing the block to a momentary halt. Assuming air resistance is negligible; from what height (in cm) above the compressed spring was the block dropped?

7. A 0.650-kg mass oscillates according to the equation where $x$ is in meters and is in seconds.

   \[ y = (0.25 \text{ m}) \sin(4.70t) \]

   Determine the total energy, the kinetic energy, and the potential energy when $y$ is 15 cm.
8. Given 

\[ y = 0.5 \cos(3\pi t) \]

where \( x \) is in meter and \( t \) is in second.

a. Calculate the time required by a body to move from the equilibrium position to a point 0.25 m away from it.

b. Calculate the velocity and acceleration of the body at \( t = 0.05 \) s.

c. Calculate the velocity and acceleration of the body at a distance 0.05 m from the equilibrium position.

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**Graphs of Simple Harmonic Motion**

1. An 0.80 kg object is attached to one end of a spring, as in Figure, and the system is set into simple harmonic motion. The displacement \( y \) of the object as a function of time is shown in the drawing.

   With the aid of these data, determine

   a. the amplitude \( A \) of the motion
   b. the angular frequency
   c. the spring constant \( k \)
   d. the speed of the object at \( t = 1.0 \) s
   e. the magnitude of the object’s acceleration at \( t = 1.0 \) s

2. The graph shows the SHM acceleration-time graph of a 0.5 kg mass attached to a spring on a smooth horizontal surface. By using the graph determine

   a. the spring constant
   b. the amplitude of oscillation
   c. the equation of displacement \( y \) varies with time, \( t \).

3. Sketch the \( y-t \) graph for equation

   a. \( y = 3 \sin(4\pi t) \)
   b. \( y = 3 \sin\left(4\pi t + \frac{\pi}{2}\right) \)
   c. \( y = 3 \sin(4\pi t - \pi) \)
   d. \( y = -3 \sin(4\pi t) \)
   e. \( y = 3 \cos(4\pi t) \)

4. A particle executes SHM with amplitude 0.05 cm and period 12 s. Its displacement at \( t = 0 \) is zero. Sketch and label the

   a. displacement-time graph
   b. velocity-time graph
   c. acceleration-time graph
## Period of Simple Harmonic Motion

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<tr>
<td>1.</td>
<td>How long must a simple pendulum be if it is to make exactly one swing per second? Given one complete oscillation takes exactly 2.0 s.</td>
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<tr>
<td>2.</td>
<td>Your grandfather clock’s pendulum has a length of 0.9930 m. If the clock runs slow and loses 21 s per day, how should you adjust the length of the pendulum?</td>
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</tbody>
</table>
| 3. | A 200 g block connected to a light spring for which the force constant is 5.00 N m\(^{-1}\) is free to oscillate on a frictionless, horizontal surface. The block is displaced 5.00 cm from equilibrium and released from rest.  
   a. Find the period of its motion  
   b. Determine the maximum speed of the block  
   c. What is the maximum acceleration of the block?  
   d. Express the position, velocity, and acceleration as functions of time in SI units. |
| 4. | You attach an object to the bottom end of a hanging vertical spring. It hangs at rest after extending the spring 18.3 cm. You then set the object vibrating. Determine the period of its motion. |
| 5. | A 950 kg car strikes a huge spring at a speed of 25 m s\(^{-1}\) compressing the spring 4.0 m.  
   a. What is the spring stiffness constant of the spring?  
   b. How long is the car in contact with the spring before it bounces off in the opposite direction? |